Pergamon

Int. J. Solids Structures Vol. 33, No. 27, pp. 4065–4078, 1996

0020–7683/96 $15.00 + .00

0020–7683(95)00225–1

A COMPLETE SET OF COROLLARIES OF THE RECIPROCAL THEOREMS IN ELASTICITY

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(*Received* 7 *July* 1994; *in revised form* 6 *September* 1995)

Abstract—This work presents a complete set of corollaries derived from the classical reciprocal theorems in the linear theory of elasticity. It is shown that the strain energy required to produce a given effect (reactive generalised force or displacement) at a certain position A of a linearly elastic body or structure by means of the application of the corresponding dual action (imposed generalised displacement or force, respectively) at another position B, attains its minimum when A coincides with B. This fact turns out to be a simple and general property of any linearly elastic model and from a qualitative point of view it can be related to the well-known local perturbation principle of Boussinesq. The assertions made can prove themselves useful in the interpretation of terms arising in several engineering problems, like boundary elements analyses or structural monitoring procedures.

1. INTRODUCTION

The study of the effects yielded by the application of a system of forces to a portion of a linearly elastic and isotropic solid body is certainly a very well trodden field in the classical theory of elasticity. Nevertheless, even nowadays, when an enormous amount of advanced numerical procedures is promptly available for any kind of structural analysis, every simple theoretical tool capable of providing a quick and reliable characterization of a solution can still be extremely valuable to engineers and researchers. In fact, engineers often have to deal with rather complicated problems, although the information they require is relatively restricted. Here lies the importance of stating simple behavioural properties, a task which has been extensively tackled in many classic works (e.g. Signorini, 1933; Diaz and Greenberg, 1948; Synge, 1950) and some noteworthy treatises (e.g. Villaggio, 1977).

Among all these assertions an important place is occupied by the local perturbation principle of Boussinesq (Boussinesq, 1885), which includes Saint-Venant's principle of the elastic equivalence of statically equipollent systems of load (Saint-Venant, 1855), this latter more often used in problems relating to bars and plates.

Historically, Boussinesq's statement derived from the observations made on some general solutions of problems of elastic equilibrium which had been obtained by means of the potential theory. In these cases the displacement due to a distribution of force having a finite resultant for a small volume varies inversely as the distance and that due to forces having zero resultant for the small volume varies inversely as the square of the distance, and directly as the linear dimension of the small volume. Therefore, one can conclude that the strain produced at a distance, by forces applied locally, depends upon the resultant of the forces and is practically independent of the mode of distribution of the forces which are statically equivalent to this resultant (Love, 1927). This means that the effects of the mode of distribution of the forces are practically confined to a comparatively small portion of the body near to the place of application of the forces. These effects were defined by Boussinesq "local perturbations".

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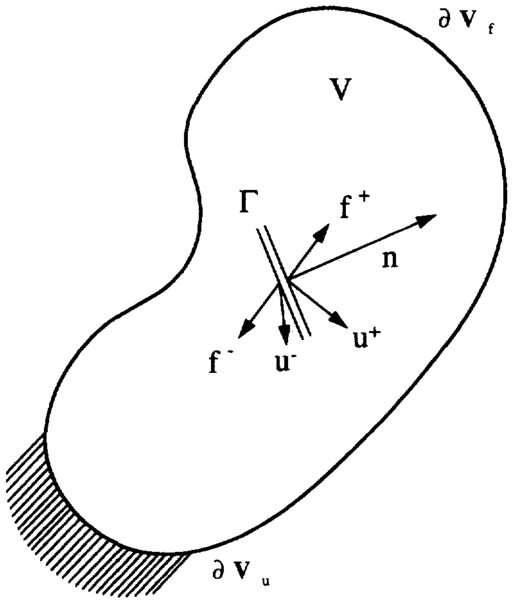

Fig. 1.

The validity and the value of this line of reasoning can be verified in a large number of structural problems and has been object of several subsequent studies (Zanaboni, 1937; von Mises, 1945; Sternberg, 1954).

Of course, other statements of the same kind can be established in the linear theory of elasticity, by means of classical tools.

In this paper we show that the strain energy required to produce a given effect (reactive generalised force or displacement) at a certain position A of a linearly elastic body or structure by means of the application of the corresponding dual action (imposed generalised displacement or force, respectively) at another position B, always attains its minimum when A coincides with B. The proof relies upon the extended version of the reciprocal theorem of Betti (Betti, 1872), where the extensions to the case of imposed discontinuities in the displacement field are essentially due to V. Volterra (Volterra, 1907) and G. Colonnetti (Colonnetti, 1912 and 1915).

Therefore, we derive a simple but complete set of corollaries of these classic reciprocal theorems. These statements can be regarded as strict relatives of the Boussinesq's principle in the sense that they show how the strain energy tends to increase when a certain effect has to be produced by means of a cause which is located further and further from the point of measure.

A subset of these corollaries has recently been noticed by some of the present authors (Minutolo and Nunziante, 1993) and is implicitly included in this much more general formulation.

For the sake of clarity, first we will demonstrate the general formulation of the above mentioned set of corollaries for elastic bodies, then we will examine the case of structural systems. Finally, we will give some highlights of the possible applications of these statements by identifying the physical meaning of the terms arising in a particular boundary element analysis for the coupling of an elastic half-space to a generic upper structure.

2. THE GENERALISED RECIPROCAL THEOREMS IN THE LINEAR THEORY OF ELASTICITY

In order to start our discourse we make reference to a linearly elastic body occupying a regular region V with smooth boundary ∂V of the three-dimensional Euclidean space (referred to a Cartesian orthogonal co-ordinate system $\boldsymbol{x} = (x_1, x_2, x_3)$), as shown in Fig. 1. For the sake of simplicity we will consider ∂V to be partitioned into ∂V_f and ∂V_u, so that $\partial V_f = \partial V / \partial V_u$.

We will take into consideration the usual kinds of external actions: tractions $\bar{f}_i$ assigned on ∂V_f, displacements $\bar{u}_i$ assigned on ∂V_u, body forces $\bar{b}_i$ and imposed strains $\bar{\vartheta}_{ij}$ over V.

Moreover we will take into consideration displacement discontinuities as further external actions. Let Γ be an arbitrary plain surface passing through a generic point $\boldsymbol{x}$, where its unit normal vector is $\mathbf{n}$. We will call a displacement discontinuity on Γ a displacement jump across Γ defined as $\mathbf{d}^n = \mathbf{u}^{n+} - \mathbf{u}^{n-}$. In this case, on account of the condition of equilibrium

$$f_i^+ = -f_i^- = -f_i \tag{1}$$

the work per surface unit around a point $\mathbf{x}$ on Γ is

$$f_i^- u_i^- + f_i^+ u_i^+ = -f_i d_i. \tag{2}$$

For linearly elastic body subjected to all these kinds of actions the generalised reciprocal theorem holds true on account of the divergence theorem and of the symmetry of the elastic tensor (Betti, 1872; Volterra, 1907; Colonnetti, 1912 and 1915). Therefore, given two elastic states $\mathscr{E} = (\bar{f}_j, \bar{u}_j, \bar{b}_j, \bar{\vartheta}_{ij}, \bar{d}_j^n, u_j, \varepsilon_{ij}, \sigma_{ij})$ and $\mathscr{E}^* = (\bar{f}_j^*, \bar{u}_j^*, \bar{b}_j^*, \bar{\vartheta}_{ij}^*, \bar{d}_j^{n*}, u_j^*, \varepsilon_{ij}^*, \sigma_{ij}^*)$, which both satisfy the governing equations of the linearly elastic boundary value problem, i.e.

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{in} \quad V \tag{3}$$

$$u_i = \bar{u}_i \quad \text{on} \quad \partial V_u \tag{4}$$

$$\sigma_{ij,j} + \bar{b}_i = 0 \quad \text{in} \quad V \tag{5}$$

$$\sigma_{ij} n_j = \bar{f}_i \quad \text{on} \quad \partial V_f \tag{6}$$

$$\varepsilon_{ij} = C_{ijhk}\sigma_{hk} + \bar{\vartheta}_{ij} \quad \text{in} \quad V \tag{7}$$

we can write

$$\int_V \bar{b}_i u_i^* \,\mathrm{d}V + \int_V \sigma_{ij}^* \bar{\vartheta}_{ij} \,\mathrm{d}V + \int_{\partial V_f} \bar{f}_i u_i^* \,\mathrm{d}S + \int_{\partial V_u} f_i \bar{u}_i^* \,\mathrm{d}S + \int_{\Gamma^*} f_i \bar{d}_i^* \,\mathrm{d}S$$
$$= \int_V \bar{b}_i^* u_i \,\mathrm{d}V + \int_V \sigma_{ij} \bar{\vartheta}_{ij}^* \,\mathrm{d}V + \int_{\partial V_f} \bar{f}_i^* u_i \,\mathrm{d}S + \int_{\partial V_u} f_i^* \bar{u}_i \,\mathrm{d}S + \int_{\Gamma} f_i^* \bar{d}_i \,\mathrm{d}S \tag{8}$$

where, of course, $\bar{f}_i$ and $\bar{f}_i^*$ are given tractions on ∂V_f, while f_i and f_i^* are reactive forces on ∂V_u and on Γ and Γ^*, yielded by means of the relationship

$$\sigma_{ij} n_j = f_i. \tag{9}$$

The generalised reciprocal theorem can be stated also in the presence of concentrated actions (Sternberg and Eubanks, 1955; Turteltaub and Sternberg 1968; Crouch and Starfield, 1983; Brebbia *et al.*, 1984). However, it is well known that this is quite a delicate matter and in most cases it requires a detailed and thorough mathematical treatment. Since the purpose of this paper consists of stating some simple but relevant properties of the linearly elastic states, we will sketch a line of reasoning which applies, where the problem makes sense, also to the case of concentrated actions and refer the interested reader to the above cited references.

Hence, we will introduce a concentrated body force B_α acting at the point $\xi \in V$ in the direction of the reference axis α as yielded by a limit procedure of the type

$$B_\alpha(\xi) = \lim_{\delta V \to 0} \int_{\delta V} b_\alpha(\xi')\,\mathrm{d}V \tag{10}$$

where δV is a volume neighbourhood of $\boldsymbol{\xi}$ out of which b_α is null and provided the limit is performed by keeping the integral constant.

In a similar way we can introduce the concept of a concentrated imposed strain $\Theta_{\alpha\beta}$ acting at the point $\boldsymbol{\eta} \in V$

$$\Theta_{\alpha\beta}(\boldsymbol{\eta}) = \lim_{\delta V \to 0} \int_{\delta V} \vartheta_{\alpha\beta}(\boldsymbol{\eta}')\,\mathrm{d}V \tag{11}$$

of a concentrated boundary force F_α acting at the point $\gamma \in \partial V_f$ in the direction of the reference axis α

$$F_\alpha(\gamma) = \lim_{\delta \partial V_f \to 0} \int_{\delta \partial V_f} f_\alpha(\gamma')\,\mathrm{d}S \tag{12a}$$

of a concentrated boundary displacement U_α acting at the point $\boldsymbol{\chi} \in \partial V_u$ in the direction of the reference axis α

$$U_\alpha(\boldsymbol{\chi}) = \lim_{\delta \partial V_u \to 0} \int_{\delta \partial V_u} f_\alpha(\boldsymbol{\chi}')\,\mathrm{d}S \tag{12b}$$

and finally that of a concentrated displacement discontinuity D^n_α of the αth displacement component acting at the point $\zeta \in V$ orthogonally to the oriented direction $\mathbf{n}$

$$D^n_\alpha(\zeta) = \lim_{\Gamma \to 0} \int_\Gamma d^n_\alpha(\zeta')\,\mathrm{d}S. \tag{13}$$

Under suitable hypotheses of regular properties and provided the domains $\mathfrak{R}$ and $\mathfrak{R}^*$, which are finite sets of (boundary or interior) points of V, are disjoint, we can write the (8) in the form

$$\begin{aligned}\sum_{\xi\in\mathfrak{R}} B_\alpha(\xi)u^*_\alpha(\xi) + \sum_{\eta\in\mathfrak{R}} \sigma^*_{\alpha\beta}(\boldsymbol{\eta})\Theta_{\alpha\beta}(\boldsymbol{\eta}) + \sum_{\gamma\in\mathfrak{R}} F_\alpha(\gamma)u^*_\alpha(\gamma)\\ + \sum_{\chi\in\mathfrak{R}^*} f_\alpha(\boldsymbol{\chi})U^*_\alpha(\boldsymbol{\chi}) + \sum_{\zeta\in\mathfrak{R}^*} f^n_\alpha(\zeta)D^{n^*}_\alpha(\zeta) = \sum_{\xi\in\mathfrak{R}^*} B^*_\alpha(\xi)u_\alpha(\xi) + \sum_{\eta\in\mathfrak{R}^*} \sigma_{\alpha\beta}(\boldsymbol{\eta})\Theta^*_{\alpha\beta}(\boldsymbol{\eta})\\ + \sum_{\gamma\in\mathfrak{R}^*} F^*_\alpha(\gamma)u_\alpha(\gamma) + \sum_{\chi\in\mathfrak{R}} f^*_\alpha(\boldsymbol{\chi})U_\alpha(\boldsymbol{\chi}) + \sum_{\zeta\in\mathfrak{R}} f^{m^*}_\alpha(\zeta)D^n_\alpha(\zeta)\end{aligned} \tag{14}$$

which represent, at least from a formal point of view, the expression of the generalised reciprocal theorem for concentrated actions.

3. A COMPLETE SET OF ENERGY EXTREMUM PRINCIPLES

In the present section we are going to derive a complete set of energy extremum principles as corollaries of the generalised reciprocal theorems previously stated.

For the sake of clarity we divide the actions applied to our linearly elastic body into two classes: the class of the applied generalised forces $\mathscr{F}$ and the class of the imposed generalised displacements $\mathscr{D}$. Moreover we will consider $\mathscr{F}^C$ and $\mathscr{D}^C$ to be the classes comprising the *work conjugates* of the elements of $\mathscr{F}$ and $\mathscr{D}$, respectively.

Let us consider a traction field $\bar{\boldsymbol{f}} = (\bar{f}_1, \bar{f}_2, \bar{f}_3)$ assigned on a neighbourhood $\delta\partial V_f(P)$ of a point P belonging to ∂V_f. We assume

$$\bar{\mathbf{f}}(P) = \int_{\delta\partial V_f(P)} \bar{f}\,\mathrm{d}S \tag{15}$$

to be the generalised force applied at P ($\bar{\mathbf{f}}(P) \in \mathscr{F}$).

The corresponding generalised displacement $\mathbf{u}(P) \in \mathscr{F}^C$ can be defined by means of the following relationships

$$\bar{\mathbf{f}}(P)\cdot\mathbf{u}(P) = \int_{\delta\partial V_f(P)} \bar{f}\cdot\boldsymbol{u}\,\mathrm{d}S \tag{16}$$

$$\mathbf{u}(P) = a\int_{\delta\partial V_f(P)} \boldsymbol{u}\,\mathrm{d}S \tag{17}$$

where $\boldsymbol{u}$ is the displacement field on $\delta\partial V_f(P)$. The symbol $\cdot$ stands for the usual inner product between vector fields and a is a scalar.

In the same manner we can define the generalised displacement $\bar{\mathbf{u}}(Q) \in \mathscr{D}$. In fact, if $\bar{\boldsymbol{u}} = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$ is the imposed displacement field on a neighbourhood $\delta\partial V_u(Q)$ of a point $Q \in \partial V_u$, we can assume

$$\bar{\mathbf{u}}(Q) = \int_{\delta\partial V_u(Q)} \bar{\boldsymbol{u}}\,\mathrm{d}S. \tag{18}$$

In turn, the corresponding generalised reactive force $\mathbf{f}(Q) \in \mathscr{D}^C$ can be defined by means of the following relationships

$$\mathbf{f}(Q)\cdot\bar{\mathbf{u}}(Q) = \int_{\delta\partial V_u(Q)} f\cdot\bar{\boldsymbol{u}}\,\mathrm{d}S \tag{19}$$

$$\mathbf{f}(Q) = b\int_{\delta\partial V_u(Q)} f\,\mathrm{d}S \tag{20}$$

where, of course, f is the reactive traction field on $\delta\partial V_u(Q)$ and b is a scalar.

Repeating the above procedure for a volume neighbourhood $\delta V(R)$ of a point $R \in V$ in case of an applied body forces field $\bar{\boldsymbol{b}} = (\bar{b}_1, \bar{b}_2, \bar{b}_3)$, we get the definition of the generalised force $\bar{\mathbf{b}}(R) \in \mathscr{F}$

$$\bar{\mathbf{b}}(R) = \int_{\delta V(R)} \bar{\boldsymbol{b}}\,\mathrm{d}V \tag{21}$$

and that of the corresponding generalised displacement $\mathbf{u}(R) \in \mathscr{F}^C$

$$\bar{\mathbf{b}}(R)\cdot\mathbf{u}(R) = \int_{\delta V(R)} \bar{\boldsymbol{b}}\cdot\boldsymbol{u}\,\mathrm{d}V \tag{22}$$

$$\mathbf{u}(R) = c\int_{\delta V(R)} \boldsymbol{u}\,\mathrm{d}V. \tag{23}$$

Similarly, we get the expression of a generalised imposed strain $\bar{\mathbf{t}}(S) \in \mathscr{D}$ at a point $S \in V$

$$\bar{\mathbf{t}}(S) = \int_{\delta V(S)} \bar{\boldsymbol{\vartheta}}\,\mathrm{d}V \tag{24}$$

and of the corresponding reactive generalised force $\mathbf{s}(S) \in \mathscr{D}^C$

$$\mathbf{s}(S)\cdot\bar{\mathbf{t}}(S) = \int_{\delta V(S)} \boldsymbol{\sigma}\cdot\bar{\boldsymbol{\vartheta}}\,\mathrm{d}V \tag{25}$$

$$\mathbf{s}(S) = d\int_{\delta V(S)} \boldsymbol{\sigma}\,\mathrm{d}V. \tag{26}$$

As $\boldsymbol{\sigma}$, $\bar{\boldsymbol{\vartheta}}$, $\mathbf{s}$ and $\bar{\mathbf{t}}$ are meant to be second order tensors, in the formula (25) their inner product is intended as

$$\boldsymbol{\sigma}\cdot\bar{\boldsymbol{\vartheta}} = \mathrm{tr}\,(\boldsymbol{\sigma}^T\bar{\boldsymbol{\vartheta}}) = \sigma_{ij}\bar{\vartheta}_{ij}. \tag{27}$$

Finally, we can introduce the definition of an imposed generalised displacement discontinuity $\bar{\mathbf{d}}^n(T) \in \mathscr{D}$ at a point $T \in V$ with reference to the oriented direction $\mathbf{n}$

$$\bar{\boldsymbol{d}}^n(\mathbf{T}) = \int_{\Gamma(T)} \bar{\mathbf{d}}^n\,\mathrm{d}S \tag{28}$$

and of the corresponding reactive force $\mathbf{f}^n(T) \in \mathscr{D}^C$

$$\mathbf{f}^n(T)\cdot\bar{\mathbf{d}}^n(T) = \int_{\Gamma(T)} \boldsymbol{f}^n\cdot\bar{\boldsymbol{d}}^n\,\mathrm{d}S \tag{29}$$

$$\mathbf{f}^n(T) = e\int_{\Gamma(T)} \boldsymbol{f}^n\,\mathrm{d}S. \tag{30}$$

At this point we can introduce the first of the above mentioned energy extremum principles. Let us consider an elastic body subjected to a generalised force $\bar{\mathbf{f}} \in \mathscr{F}$ which is applied at a point $P_1 \in \partial V_f$ and let $\mathbf{u}(P_1) \in \mathscr{F}^C$ be the corresponding generalised displacement. Moreover, let consider another point $P_2 \in \partial V_f$ and its neighbourhood $\delta\,\partial V_f(P_2)$, such that $\delta\,\partial V_f(\mathbf{P}_2) \cap \delta\,\partial V_f(P_1) = \varnothing$, where $\delta\,\partial V_f(P_1)$ is the domain of definition of the generalised force $\bar{\mathbf{f}}(P_1)$.

In virtue of the linearity of the relationships (3)–(7), which hold the boundary value problem of elastostatics, and of the definitions (15)–(17), we can normally find a set of generalised forces $\bar{\mathbf{f}}^i(P_2)$ such that the generalised displacement $\mathbf{u}$ in P_1 due to any of the $\bar{\mathbf{f}}^i(P_2)$ is equal to the generalised displacement $\mathbf{u}$ in P_1 due to $\bar{\mathbf{f}}(P_1)$, i.e. $\mathbf{u}^i(P_1, P_2) = \mathbf{u}(P_1, P_1)$.

In virtue of Clapeyron's theorem (Lamé, 1852), the strain energy associated with the elastic state $\mathscr{E}(P_1)$, corresponding to the generalised force $\bar{\mathbf{f}}(P_1)$ applied at P_1, is given by

$$W_{P_1} = \tfrac{1}{2}\bar{\mathbf{f}}(P_1)\cdot\mathbf{u}(P_1, P_1) > 0 \tag{31}$$

while the strain energy associated to any of the elastic states $\mathscr{E}^i(P_2)$, corresponding to each of the above defined generalised forces $\bar{\mathbf{f}}^i(P_2)$ applied at P_2 is

$$W^i_{P_2} = \tfrac{1}{2}\bar{\mathbf{f}}^i(P_2)\cdot\mathbf{u}^i(P_2, P_2) > 0. \tag{32}$$

If we now take into consideration the strain energy associated to an elastic state corresponding to the application of both $\bar{\mathbf{f}}(P_1)$ at P_1 and $-\bar{\mathbf{f}}^i(P_2)$ at P_2, we have

$$W^{i}_{P_1+P_2} = \tfrac{1}{2}\{\bar{\mathbf{f}}(P_1)\cdot[\mathbf{u}(P_1,P_1)-\mathbf{u}^{i}(P_1,P_2)]-\bar{\mathbf{f}}^{i}(P_2)\cdot[-\mathbf{u}^{i}(P_2,P_2)+\mathbf{u}(P_2,P_1)]\} > 0. \tag{33}$$

By hypothesis it is

$$\mathbf{u}^{i}(P_1,P_2) = \mathbf{u}(P_1,P_1) \tag{34}$$

and in virtue of the reciprocal theorem (8) and of this relationship we have

$$-\bar{\mathbf{f}}^{i}(P_2)\cdot\mathbf{u}(P_2,P_1) = -\bar{\mathbf{f}}(P_1)\cdot\mathbf{u}^{i}(P_1,P_2) = -\bar{\mathbf{f}}(P_1)\cdot\mathbf{u}(P_1,P_1). \tag{35}$$

Consequently, we can write (33) in the form

$$W^{i}_{P_1+P_2} = \tfrac{1}{2}[\bar{\mathbf{f}}^{i}(P_2)\cdot u^{i}(P_2,P_2)-\bar{\mathbf{f}}(P_1)\cdot\mathbf{u}(P_1,P_1)] = W^{i}_{P_2}-W_{P_1} > 0. \tag{36}$$

We can therefore conclude that in a linearly elastic body the strain energy required to produce a certain generalised displacement $\mathbf{u}(P_1)$ at a point P_1, associated with an assigned generalised force $\bar{\mathbf{f}}(P_1)$ at the same point, constitutes a lower bound for all the values of the strain energy required to produce the same generalised displacement $\mathbf{u}(P_1)$ by means of any generalised force $\bar{\mathbf{f}}^{i}(P_2)$ applied at a different point P_2, provided $\delta\,\partial V_f(P_2)\cap\delta\,\partial V_f(P_1) = \varnothing$.

It is straightforward to verify that the same conclusion can be reached for any element belonging to the class of the applied generalised forces $\mathscr{F}$ with regard to the corresponding generalised displacement belonging to $\mathscr{F}^C$.

On the same basis this energy extremum property holds true for the imposed generalised displacements belonging to the class $\mathscr{D}$ in connection with their work conjugates belonging to $\mathscr{D}^c$.

For example we can take into account a generalised displacement $\bar{\mathbf{u}}(Q_1)\in\mathscr{D}$ applied at a point $Q_1\in\partial V_u$. Let $\bar{\mathbf{f}}(Q_1)\in\mathscr{D}^c$ be the corresponding generalised reactive force. As before, let us consider another point $Q_2\in\partial V_u$ and its neighbourhood $\delta\,\partial V_u(Q_2)$, such that $\delta\,\partial V_u(Q_2)\cap\delta\,\partial V_u(Q_1) = \varnothing$. Of course, $\delta\,\partial V_u(Q_1)$ is meant to be the domain of definition of the generalised displacement $\bar{\mathbf{u}}(Q_1)$.

Also in this case we can normally find a set of generalised displacements $\bar{\mathbf{u}}^{j}(Q_2)$ such the generalised reactive force $\mathbf{f}$ in Q_1 due to any of the $\bar{\mathbf{u}}^{j}(Q_2)$ is equal to the generalised reactive force $\mathbf{f}$ in Q_1 due to $\bar{\mathbf{u}}(Q_1)$, i.e. $\mathbf{f}^{j}(Q_1,Q_2) = \mathbf{f}(Q_1,Q_1)$.

Once again, in virtue of Clapeyron's theorem, the strain energy associated with the elastic states $\mathscr{E}(Q_1)$ and $\mathscr{E}^{j}(Q_2)$, corresponding, respectively, to the generalised displacement $\bar{\mathbf{u}}(Q_1)$ applied at Q_1 and to the generalised displacement $\bar{\mathbf{u}}^{j}(Q_2)$ applied at Q_2, is given by

$$W_{Q_1} = \tfrac{1}{2}\bar{\mathbf{u}}(Q_1)\cdot\mathbf{f}(Q_1,Q_1) > 0 \tag{37}$$

$$W^{j}_{Q_2} = \tfrac{1}{2}\bar{\mathbf{u}}^{j}(Q_2)\cdot\mathbf{f}(Q_2,Q_2) > 0. \tag{38}$$

Taking into consideration the strain energy associated with an elastic state corresponding to the application of both $\bar{\mathbf{u}}(Q_1)$ at Q_1 and $-\bar{\mathbf{u}}^{j}(Q_2)$ at Q_2, we have

$$W^{j}_{Q_1+Q_2} = \tfrac{1}{2}\{\bar{\mathbf{u}}(Q_1)\cdot[\mathbf{f}(Q_1,Q_1)-\mathbf{f}^{j}(Q_1,Q_2)] \\ -\bar{\mathbf{u}}^{j}(Q_2)\cdot[-\mathbf{f}^{j}(Q_2,Q_2)+\mathbf{f}(Q_2,Q_1)] > 0. \tag{39}$$

As by hypothesis it is

$$\mathbf{f}^j(Q_1, Q_2) = \mathbf{f}(Q_1, Q_1) \tag{40}$$

in virtue of the reciprocal theorem (8) and of this relationship we also have

$$-\bar{\mathbf{u}}^j(Q_2)\cdot\mathbf{f}(Q_2, Q_1) = -\bar{\mathbf{u}}(Q_1)\cdot\mathbf{f}^j(Q_1, Q_2) = -\bar{\mathbf{u}}(Q_1)\cdot\mathbf{f}(Q_1, Q_1). \tag{41}$$

Finally, we can write

$$W^j_{Q_1+Q_2} = \tfrac{1}{2}[\bar{\mathbf{u}}^j(Q_2)\cdot\mathbf{f}^j(Q_2, Q_2) - \bar{\mathbf{u}}(Q_1)\cdot\mathbf{f}(Q_1, Q_1)] = W^j_{Q_2} - W_{Q_1} > 0 \tag{42}$$

which proves that in a linearly elastic body the strain energy required to produce a certain generalised reactive force $\mathbf{f}(Q_1)$ as a point Q_1, associated with an imposed generalised displacement $\bar{\mathbf{u}}(Q_1)$ at the same point, constitutes a lower bound for all the values of the strain energy required to produce the same generalised reactive force $\mathbf{f}(Q_1)$ by means of any generalised displacement $\bar{\mathbf{u}}^j(Q_2)$ imposed at a different point Q_2, provided $\delta\partial V_u(Q_2) \cap \delta\partial V_u(Q_1) = \varnothing$.

Quite obviously, the same happens for any imposed generalised displacements belonging to the class $\mathscr{D}$ and its relative work conjugate belonging to $\mathscr{D}^C$.

As a result of what we have up to now shown, we can state the following general energy extremum principle:

> In a linearly elastic body the strain energy required to produce at a point A a generalised effect conjugate to a cause belonging to the class of the generalised forces $\mathscr{F}$ or to the class of the generalised displacements $\mathscr{D}$, applied at the same point A, constitutes a lower bound for all the values of the strain energy required to produce the same generalised effect by means of any corresponding causes applied at a different point B, provided the regions of definition of the applied actions are disjoint.

At this point of our discourse it is worth making some remarks.

First of all it appears clear that the introduced definitions of work conjugates of the generalised applied forces or displacements rely upon the imposed forces or displacement fields themselves. In fact the scalar quantities a, b, c, d and e which respectively appears in (17), (20), (23), (26) and (30) can be determined by means of the work relationships (16), (19), (22), (25) and (29).

It would appear natural to expect that, in the limit, if the generic loaded neighbourhood tends to zero, i.e. $\delta V \to 0$ or $\delta\partial V \to 0$, and the value of the resulting load is kept constant, we will have

$$\lim_{\partial V \to 0} a = 1, \ldots, \lim_{\partial V \to 0} e = 1 \tag{43}$$

and therefore the elements of the classes $\mathscr{F}^C$ and $\mathscr{D}^C$ will result independent from the applied force or displacement fields.

This is certainly the case in the theory of the structures, where any generic solution is averaged by definition in terms of suitable characteristics, as we are going to show in the next section.

However, the above stated energy principles require particular care when dealing with concentrated loads and imposed point displacements in the mechanics of continua. In fact, in order to reach the same conclusions as in the case of generalised forces and displacements, on one hand we have to make reference to the reciprocal theorem in the form (14), on the other we have to postulate some regular properties of the linear elastostatic equations and, above all, the existence of a unique and finite value of the strain energy of the body corresponding to the application of a concentrated action. We must have, for example,

$$F_\alpha(\gamma)u_\alpha(\gamma) = \int_V \sigma_{ij}\varepsilon_{ij}\,\mathrm{d}V = \int_V \varepsilon_{hk}D_{hkij}\varepsilon_{ij}\,\mathrm{d}V = |k| \quad k\in\mathbb{R} \tag{44}$$

which means that the displacement gradient of the solution for concentrated load must be square integrable in V, i.e. $u_{i,j}\in L_2(V)$. Unfortunately this requirement, which could seem trivial from a physical point of view, is not fulfilled by the majority of the solutions for concentrated loads available in the literature (e.g. Love, 1927; Sternberg and Eubanks, 1955; Brebbia, Telles and Wroebel, 1984). Nevertheless it seems likely to the present authors that in case these basic requirements are satisfied (and the problem assumes a precise meaning from a physical point of view), the procedure outlined leads to an energy principle formally similar to the previously introduced ones.

Before concluding this section, we are finally going to show how these energy extremum principles hold true also in the case of applied forces and displacements at the same time.

Let us consider our body subjected to both a generalised force $\bar{\mathbf{f}}\in\mathscr{F}$ applied at a point $P_1\in\partial V_f$ and a generalised displacement $\bar{\mathbf{u}}\in\mathscr{D}$ applied at a point $Q_1\in\partial V_u$. As before, let us take into consideration another point $P_2\in\partial V_f$ with a related neighbourhood $\delta\,\partial V_f(P_2)$, as well as another point $Q_2\in\partial V_u$ with a related neighbourhood $\delta\,\partial V_u(Q_2)$. We require that

$$\delta\,\partial V_f(P_2)\cap\delta\,\partial V_f(P_1)\cap\delta\,\partial V_u(Q_2)\cap\delta\,\partial V_u(Q_1) = \varnothing \tag{45}$$

where $\delta\,\partial V_f(P_1)$ is the domain of definition of the generalised force $\bar{\mathbf{f}}(P_1)$ and $\delta\,\partial V_u(Q_1)$ is the domain of definition of the generalised displacement $\bar{\mathbf{u}}(Q_1)$.

Let $\bar{\mathbf{f}}^i(P_2)$ be a set of generalised forces defined on $\delta\,\partial V_f(P_2)$ and $\bar{\mathbf{u}}^j(Q_2)$ a set of generalised displacements defined on $\delta\,\partial V_u(Q_2)$ such that

$$\mathbf{u}^i(P_1,P_2)+\mathbf{u}^j(P_1,Q_2) = \mathbf{u}(P_1,P_1)+\mathbf{u}(P_1,Q_1) \tag{46}$$

$$\mathbf{f}^j(Q_1,Q_2)+\mathbf{f}^i(Q_1,P_2) = \mathbf{f}(Q_1,Q_1)+\mathbf{f}(Q_1,P_1). \tag{47}$$

The strain energy associated with the elastic state $\mathscr{E}(P_1+Q_1)$, corresponding to the application of both $\bar{\mathbf{f}}(P_1)$ and $\bar{u}(Q_1)$ is given by

$$\begin{aligned} W_{P_1+Q_1} &= \tfrac{1}{2}\{\bar{\mathbf{f}}(P_1)\cdot[\mathbf{u}(P_1,P_1)+\mathbf{u}(P_1,Q_1)]+\bar{\mathbf{u}}(Q_1)\cdot[\mathbf{f}(Q_1,Q_1)+\mathbf{f}(Q_1,P_1)]\} \\ &= W_{P_1}+W_{Q_1}+\bar{\mathbf{f}}(P_1)\cdot\mathbf{u}(P_1,Q_1) = W_{P_1}+W_{Q_1}+\bar{\mathbf{u}}(Q_1)\cdot\mathbf{f}(Q_1,P_1) > 0 \end{aligned} \tag{48}$$

in virtue of (8).

Similarly, the strain energy associated to the state $\mathscr{E}^{ij}(P_2+Q_2)$ is given by

$$\begin{aligned} W^{ij}_{P_2+Q_2} &= \tfrac{1}{2}\{\bar{\mathbf{f}}^i(P_2)\cdot[\mathbf{u}^i(P_2,P_2)+\mathbf{u}^j(P_2,Q_2)]+\bar{\mathbf{u}}^j(Q_2)\cdot[\mathbf{f}^j(Q_2,Q_2)+\mathbf{f}^i(Q_2,P_2)]\} \\ &= W^i_{P_2}+W^j_{Q_2}+\bar{\mathbf{f}}^i(P_2)\cdot\mathbf{u}^j(P_2,Q_2) = W^i_{P_2}+W^j_{Q_2}+\bar{\mathbf{u}}^j(Q_2)\cdot\mathbf{f}^i(Q_2,P_2) > 0. \end{aligned} \tag{49}$$

Applying to the body under consideration all the actions $\bar{\mathbf{f}}(P_1)$, $\bar{\mathbf{u}}(Q_1)$, $-\bar{\mathbf{f}}^i(P_2)$ and $-\bar{\mathbf{u}}^j(Q_2)$ at the same time, we have

$$\begin{aligned} W^{ij}_{P_1+Q_1-P_2+Q_2} = \tfrac{1}{2}\{&\bar{\mathbf{f}}(P_1)\cdot[\bar{\mathbf{u}}(P_1,P_1)+\mathbf{u}(P_1,Q_1)-\mathbf{u}^i(P_1,P_2)-\mathbf{u}^j(P_1,Q_2)] \\ &+\bar{\mathbf{u}}(Q_1)\cdot[\mathbf{f}(Q_1,P_1)+\mathbf{f}(Q_1,Q_1)-\mathbf{f}^i(Q_1,P_2)-\mathbf{f}^j(Q_1,Q_2)] \\ &-\bar{\mathbf{f}}^i(P_2)\cdot[\mathbf{u}(P_2,P_1)+\mathbf{u}(P_2,Q_1)-\mathbf{u}^i(P_2,P_2)-\mathbf{u}^j(P_2,Q_2)] \\ &-\bar{\mathbf{u}}^j(Q_2)\cdot[\mathbf{f}(Q_2,P_1)+\mathbf{f}(Q_2,Q_1)-\mathbf{f}^i(Q_2,P_2)-\mathbf{f}^j(Q_2,Q_2)]\} > 0. \end{aligned} \tag{50}$$

Once again, making reference to the reciprocal theorem (8) and in virtue of the relationships (46), (47) and (49), we can write

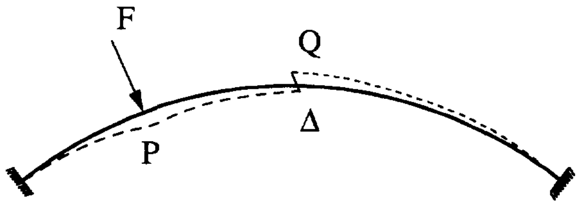

Fig. 2.

$$
\begin{aligned}
&\tfrac{1}{2}\{-\bar{\mathbf{f}}^i(P_2)\cdot\mathbf{u}(P_2,P_1)-\bar{\mathbf{f}}^i(P_2)\cdot\mathbf{u}(P_2,Q_1)-\bar{\mathbf{u}}^j(Q_2)\cdot\mathbf{f}(Q_2,P_1)\\
&\quad-\bar{\mathbf{u}}^j(Q_2)\cdot\mathbf{f}(Q_2,Q_1)\}=\tfrac{1}{2}\{-\bar{\mathbf{f}}(P_1)\cdot\mathbf{u}(P_1,P_2)\\
&\quad-\bar{\mathbf{u}}(Q_1)\cdot\mathbf{f}(Q_1,P_2)-\bar{\mathbf{f}}(P_1)\cdot\mathbf{u}(P_1,Q_2)-\bar{\mathbf{u}}(Q_1)\cdot\mathbf{f}(Q_1,Q_2)\}\\
&\quad=\tfrac{1}{2}\{-\bar{\mathbf{f}}(P_1)\cdot[\mathbf{u}(P_1,P_1)+\mathbf{u}(P_1,Q_1)]-\bar{\mathbf{u}}(Q_1)\cdot[\mathbf{f}(Q_1,Q_1)+\mathbf{f}(Q_1,P_1)]\}=-W_{P_1+Q_1}.
\end{aligned}
\tag{51}
$$

Therefore from (50), on account of (48) and (51), we finally get

$$
W^{ij}_{P_1+Q_1+P_2+Q_2}=W^{ij}_{P_2+Q_2}-W_{P_1+Q_1}>0 \tag{52}
$$

which proves that in a linearly elastic body the strain energy required to produce a certain generalised displacement $\mathbf{u}(P_1)$ at a point P_1, associated to an assigned generalised force $\bar{\mathbf{f}}(P_1)$ at the same point, and a certain generalised reactive force $\mathbf{f}(Q_1)$ at a point Q_1, associated to an imposed generalised displacement $\bar{\mathbf{u}}(Q_1)$ at the same point, constitutes a lower bound for all the values of the strain energy required to produce the same generalised displacement $\mathbf{u}(P_1)$ and the same generalised reactive force $\mathbf{f}(Q_1)$ by means of any generalised forces $\bar{\mathbf{f}}^i(P_2)$ and any generalised displacements $\bar{\mathbf{u}}^j(Q_2)$ applied, respectively, at different points P_2 and Q_2, provided $\delta\,\partial V_f(P_2)\cap\delta\,\partial V_f(P_1)\cap\delta\,\partial V_u(Q_2)\cap\delta\,\partial V_u(Q_1)=\varnothing$.

As the pathway which led to (52) remains valid for any couples of elements belonging to $\mathscr{F}$ and $\mathscr{D}$, we can state the following principle in full generality:

> In a linearly elastic body the strain energy required to produce at a point A a generalised effect conjugate to a cause belonging to the class of the generalised forces $\mathscr{F}$ and at a point B a generalised effect conjugate to a cause belonging to the class of the generalised displacements $\mathscr{D}$, applied at the same points A and B, respectively, constitutes a lower bound for all the values of the strain energy required to produce the same generalised effects by means of any couples of corresponding causes applied in turn at different points C and D, provided all the regions of definition of the applied actions are disjoint.

4. THE SET OF ENERGY PRINCIPLES IN THE THEORY OF THE STRUCTURES

In this section we are going to apply the previously introduced set of energy extremum principles to the case of the systems of beams. This is a typical case of the theory of the structures and will allow us to point out the particularly simple and expressive meaning of these principles when applied to systems which are studied in terms of characteristics, i.e. suitable and *a priori* defined averages of pointwise quantities. As the line of reasoning remains exactly the same as followed in the previous section, for the sake of brevity we will consider the sole case of forces and displacement discontinuities active at the same time.

Let us consider a linearly elastic and statically indeterminate frame-structure (e.g. the one shown in Fig. 2), subjected to a concentrated force $\mathbf{F}$ at a point P of its axis and to a displacement discontinuity $\boldsymbol{\Delta}$ at another point Q. This means that in Q the following relationship is given

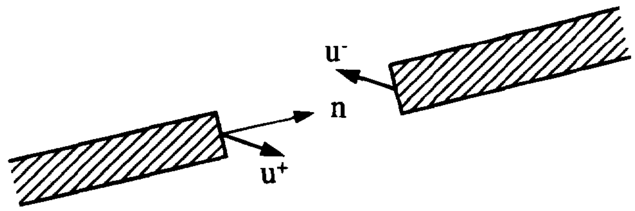

Fig. 3.

$$\mathbf{U}^+ - \mathbf{U}^- = \mathbf{\Delta} \tag{53}$$

where $\mathbf{U}^+$ is the displacement vector of the face of normal $\mathbf{n}^+$ and $\mathbf{U}^-$ the displacement vector of the opposite face, as depicted in Fig. 3. In case Q is a constrained point, $\mathbf{\Delta}$ will represent the imposed displacement of the constraint.

The displacement vector $\mathbf{U}(A)$ of a generic point A of the axis of the structure depends on both these actions and is given by

$$\mathbf{U}(A) = \mathbf{G}(A,P)\mathbf{F}(P) + \mathbf{J}(A,Q)\mathbf{\Delta}(Q) \tag{54}$$

where $\mathbf{G}(A,P)$ is the influence function matrix for the displacements of the point A on account of unit forces applied at P and $\mathbf{J}(A,Q)$ is the influence function matrix for the displacements of the same point due to unit displacement discontinuities at Q. In the same manner, as the structure under consideration is statically indeterminate, the internal force $\mathbf{R}(B)$ at a generic point B of the axis of the structure depends on both the applied force $\mathbf{F}(P)$ and the imposed discontinuity $\mathbf{\Delta}(Q)$, i.e.

$$\mathbf{R}(A) = \mathbf{L}(A,P)\mathbf{F}(P) + \mathbf{H}(A,Q)\mathbf{\Delta}(Q) \tag{55}$$

where the meaning of the influence matrices $\mathbf{L}(A,Q)$ and $\mathbf{H}(A,P)$ is evident by analogy with (54).

Of course, if B is a constrained point, the vector $\mathbf{R}(B)$ represents the reaction of the constraint on the structure.

By taking into account the reciprocal theorem (8), we immediately conclude that the following relationships hold true

$$\mathbf{G}^T(P,A) = \mathbf{G}(A,P) \quad \mathbf{H}^T(Q,B) = \mathbf{H}(B,Q) \quad \mathbf{J}^T(P,Q) = \mathbf{L}(Q,P). \tag{56}$$

For the sake of simplicity we introduce the following positions

$$\mathbf{a}_{PQ} = \begin{bmatrix} \mathbf{F}(P) \\ \mathbf{\Delta}(Q) \end{bmatrix} \quad \mathbf{r}_{AB} = \begin{bmatrix} \mathbf{U}(A) \\ \mathbf{R}(B) \end{bmatrix} \tag{57}$$

and write (54) and (55) in the compact form

$$\mathbf{r}_{AB} = \mathbf{M}_{ABPQ}\mathbf{a}_{PQ} \tag{58}$$

where the influence matrix $\mathbf{M}_{ABPQ}$ is given by

$$\mathbf{M}_{ABPQ} = \begin{bmatrix} \mathbf{G}(A,P) & \mathbf{J}(A,Q) \\ \mathbf{L}(B,P) & \mathbf{H}(B,Q) \end{bmatrix}. \tag{59}$$

Therefore we can write the strain energy of the elastic structure under analysis as

$$W_{P+Q} = \tfrac{1}{2}\mathbf{a}_{PQ}^T\mathbf{r}_{PQ} = \tfrac{1}{2}\mathbf{a}_{PQ}^T\mathbf{M}_{PQPQ}\mathbf{a}_{PQ} > 0. \quad (60)$$

On account of its definition (59), we can easily deduce that the influence function matrix $\mathbf{M}_{ABPQ}$ is symmetric and positive definite.

Hence, for any couples of points A and B we can find a unique vector $\bar{\mathbf{a}}_{AB}$, standing for an applied force at A and an imposed displacement at B, such that the related displacement $\mathbf{U}$ of P and internal force $\mathbf{R}$ at Q are exactly the same of those due to $\mathbf{F}(P)$ and $\mathbf{\Delta}(Q)$. In fact we have

$$\bar{\mathbf{a}}_{AB} = \mathbf{M}_{PQAB}^{-1}\mathbf{r}_{PQ}. \quad (61)$$

Let us now consider our structure subjected to the actions $\mathbf{a}_{PQ}$ and $-\bar{\mathbf{a}}_{AB}$. The strain energy results to be

$$W_{P+Q+A+B} = \tfrac{1}{2}[\mathbf{a}_{PQ}^T(\mathbf{M}_{PQPQ}\mathbf{a}_{PQ} - \mathbf{M}_{PQAB}\bar{\mathbf{a}}_{AB}) - \bar{\mathbf{a}}_{AB}^T(-\mathbf{M}_{ABAB}\bar{\mathbf{a}}_{AB} + \mathbf{M}_{ABPQ}\mathbf{a}_{PQ})] > 0 \quad (62)$$

and in virtue of (58) and (61) we immediately have

$$W_{P+Q+A+B} = W_{A+B} - W_{P+Q} > 0. \quad (63)$$

This allows us to state the energy extremum principle in the following form:

> In a linearly elastic and statically indeterminate frame structure, the strain energy required to produce a given displacement at a point P and a given internal force or reaction at a point Q, through the application of a force $\mathbf{F}$ and a displacement discontinuity $\mathbf{\Delta}$, attains its minimum when $\mathbf{F}$ and $\mathbf{\Delta}$ are exactly applied at P and Q, respectively.

The principle holds obviously true for several applied forces $\mathbf{F}^1, \mathbf{F}^2, \ldots, \mathbf{F}^n$ and several imposed displacement discontinuities $\mathbf{\Delta}^1, \mathbf{\Delta}^2, \ldots, \mathbf{\Delta}^n$ applied at the points $P_1, P_2, \ldots, P_n$ and $Q_1, Q_2, \ldots, Q_n$, respectively.

The results are so clear that, in the theory of the structures, where the behaviour of the elastic systems is described in terms of characteristics, the terms involved in the energy extremum principles previously illustrated assume a straightforward physical connotation that in the present case are the displacements and the forces related to each point of the axis of the frame structure, according to the classical theory of trusses and beams.

We can observe similar circumstances in the Finite-Element analyses, where the generalised displacements, as well as the generalised forces, are normally defined with reference to the nodes of the model and the solution of the problem is pursued in a finite-dimensional vector space.

5. AN APPLICATION TO A BOUNDARY ELEMENT ANALYSIS

In this final section we show a simple application of the previously examined energy extremum principles. In particular we will identify the properties of the matrix which describes the behaviour of an elastic half-space in a simple analysis of soil-structure interaction by BEM-FEM coupling (Guarracino *et al.*, 1992).

Let us consider the elastic half-space Ω and its limit plane Π. With reference to the Boussinesq-Cerruti (Love, 1927) fundamental solution and assuming vanishing body forces, we can represent the solution by means of the following Somigliana's identity

$$\delta_{ij}u_j(\xi) = \int_{\tilde{\Pi}} G_{ij}(x, \xi)t_j \, dS(x) \tag{64}$$

where $\tilde{\Pi}$ is the loaded potion of Π and t_j are the components of traction. The symbol δ_{ij} is the Kronecker delta.

If, on the contrary, the field $u(\xi)$ is prescribed on Π, (64) is to be considered as a Fredholm integral equation of the first kind in the unknown function $t(x)$.

A convenient discretization of $\tilde{\Pi}$ can be carried out by means of a mesh of non-conforming one-node rectangular and identical elements. In this case the collocation procedure of (64) in the nodes yields

$$u_j^q = \sum_p t_i^p \int_{\tilde{\Pi}^p} G_{ij}(x, \xi^q) \, dS(x) \tag{65}$$

where superscripts p and q stand for the mesh elements and the nodes of collocation, respectively.

Equation (65) can be written in the following matrix form

$$\mathbf{U} = \mathbf{H} \cdot \mathbf{T} \tag{66}$$

where vectors $\mathbf{U}$ and $\mathbf{T}$ collect the variables u_j^q and t_i^p, respectively.

As (66) can give origin to a linear algebraic system of equations, which is going to be coupled with the system derived from the Finite-Element treatment of the superior structure, it is of fundamental interest to recognise the properties of the operator $\mathbf{H}$.

Its symmetry immediately descends from the symmetry of the kernel $G_{ij}(x, \xi)$ and the geometry of the mesh. Its positive-definiteness can be ascertained by means of the previously stated energy principles. It is in fact sufficient to observe that the leading diagonal minors of $\mathbf{H}$ are all positive.

Therefore, let us consider the sub matrix $\mathbf{H}_{33}$, given by the entries common to the first three rows and columns of $\mathbf{H}$. This sub matrix is positive-definite because its non-zero entries are all positive and located on the main diagonal.

The sub matrix $\mathbf{H}_{44}$ is given by

$$\mathbf{H}_{44} = \begin{bmatrix} \mathbf{H}_{33} & \mathbf{d} \\ \mathbf{d}^T & c \end{bmatrix} \tag{67}$$

and its determinant can be written as (Noble and Daniel, 1977)

$$\det \mathbf{H}_{44} = \det \mathbf{H}_{33}(c - \mathbf{d}^T \mathbf{H}_{33}^{-1} \mathbf{d}). \tag{68}$$

It is immediately recognised that the term $\mathbf{H}_{33}^{-1}\mathbf{d}$ represents the force that must be applied on the first element in order to produce the displacement $\mathbf{d}$ of its node. The term c represents the i-th component of displacement of the node of the second element yielded by a unit force applied at the node itself. In turn, this unit force yields a displacement of the node of the first element that is again represented by the vector $\mathbf{d}$.

Therefore, on account of the previously stated energy extremum principles we have

$$\mathbf{1}c > \mathbf{d}^T \mathbf{H}_{33}^{-1} \mathbf{d} \tag{69}$$

and consequently $\det \mathbf{H}_{44} > 0$.

By iterating this procedure we can effortlessly conclude that the algebraic operator $\mathbf{H}$ is positive-definite.

6. CONCLUSIONS

The energy extremum principles illustrated in the present paper constitute an easily applicable tool in the analysis of elastic problems. Apart from their conceptual meaning and their historical relationship with Boussinesq's assumptions, their simplicity allows a quick and reliable characterization of many problems and in view of the authors it can turn out useful in several engineering issues, whenever the effects of a local action needs to be *a priori* identified or bounded.

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